Lagrangian-Hamiltonian formulation for stationary flows of some class of nonlinear dynamical systems

1993 J. Phys. A: Math. Gen. 26 L263

(http://iopscience.iop.org/0305-4470/26/5/016)

J. Phys. A: Math. Gen. 26 (1993) L263–L265. Printed in the UK

LETTER TO THE EDITOR

Lagrangian–Hamiltonian formulation for stationary flows of some class of nonlinear dynamical systems

Maciej Błaszak†
Departamento de Física, Universidade da Beira Interior, 6200 Covilhã, Portugal

Received 20 November 1992

Abstract. We construct Lagrangian and Hamiltonian functions for stationary flows of some class of Hamiltonian nonlinear dynamical systems. The method is illustrated by the example of stationary flows of Sawada-Kotera hierarchy.

Let us consider the $(2n+3)$th order Hamiltonian evolution equation

$$u_t = B(u)\frac{\delta\mathscr{H}[u]}{\delta u} \tag{1}$$

where

$$B(u)=\partial^3+au\partial+a\partial u \qquad a=\text{const} \tag{2}$$

is a Poisson operator, $\mathscr{H}[u]=\mathscr{H}(u,u_x,\ldots,u_{nx})$ is a differential function of order n and $\delta/\delta u$ is a variational derivative. Its stationary equation has the form

$$(\partial^3+au\partial+a\partial u)\frac{\delta\mathscr{H}[u]}{\delta u}=0 \tag{3}$$

and we are looking for a Lagrangian representation of (3).

Let $\gamma=\delta\mathscr{H}/\delta u$, so (3) reads

$$\gamma_{xxx}+2au\gamma_x+au_x\gamma=0. \tag{4}$$

Multiplying (4) by γ and integrating once we obtain

$$\gamma\gamma_{xx}-\tfrac{1}{2}\gamma_x^2+au\gamma^2=\tfrac{1}{8}a^2\alpha \tag{5}$$

where α is the integration constant. Introducing a new variable v

$$\gamma=-\tfrac{1}{4}av^2 \tag{6}$$

we transform (5) to the form

$$2v^3v_{xx}+auv^4=\alpha. \tag{7}$$

† On leave of absence from Physics Department of A Mickiewicz University, 60-769 Poznań, Poland.

Thus, the stationary equation (3) is equivalent to the pair of equations (6) and (7), that is

$$\frac{\delta\mathscr{H}[u]}{\delta u}+\tfrac{1}{4}av^2=0 \qquad v_{xx}+\tfrac{1}{2}auv-\frac{\alpha}{2v^3}=0. \tag{8}$$

But equations (8) come from the Euler-Lagrange equation $\delta\mathscr{L}=0$, where $\delta=(\delta/\delta u,\ \delta/\delta v)^T$ and

$$\mathscr{L}[u,v]=\mathscr{H}[u]+\tfrac{1}{4}auv^2-\tfrac{1}{2}v_x^2+\frac{\alpha}{4v^2}. \tag{9}$$

In the case of non-degenerated $\mathscr{H}[u]$, i.e. for $\partial\mathscr{H}/\partial u_{nx}\neq 0$, we can construct the canonical Hamiltonian representation of (3) in terms of so-called Ostrogradsky variables [1] through a generalized Legendre transformation [2]

$$\begin{aligned}
&q_k=u_{(k-1)x} \qquad p_k=\delta\mathscr{L}/\delta u_{kx} \qquad k=1,\ldots,n\\
&q_{n+1}=v \qquad p_{n+1}=\delta\mathscr{L}/\delta v_x=-v_x\\
&h(q,p)=-q_{n+1}p_{n+1}+q_n(q_n)_x+\sum_{k=1}^{n-1}p_kq_{k+1}-\mathscr{L}
\end{aligned} \tag{10}$$

where q_k, p_k are conjugate variables and $h(q,p)$ is a Hamiltonian function.

We illustrate our approach on the example of stationary flow of seventh-order Sawada-Kotera equation:

$$\begin{aligned}
0&=(\partial^3+au\partial+a\partial u)\frac{\delta}{\delta u}\left(\tfrac{1}{2}u_{2x}^2-\tfrac{3}{4}auu_x^2+\tfrac{1}{24}a^2u^4\right)\\
&=(\partial^3+au\partial+a\partial u)\left(u_{4x}+\tfrac{3}{2}auu_{2x}+\tfrac{1}{2}au_x^2+\tfrac{1}{6}a^2u^3\right)\\
&=u_{7x}+\tfrac{7}{2}auu_{5x}+7au_xu_{4x}+\tfrac{21}{2}au_{2x}u_{3x}+\tfrac{7}{2}a^2uu_{3x}+\tfrac{21}{2}a^2u_xu_{2x}u_{3x}+\tfrac{7}{4}a^2u_x^3\\
&\quad+\tfrac{7}{6}a^3u^3u_x.
\end{aligned} \tag{11}$$

The system (11) is equivalent to the following one

$$\left.\begin{aligned}
&u_{4x}+\tfrac{3}{2}auu_{2x}+\tfrac{3}{4}u_x^2+\tfrac{1}{6}a^2u^3+\tfrac{1}{4}av^2=0\\
&v_{2x}+\tfrac{1}{2}auv-(\alpha/2v^3)=0
\end{aligned}\right\}\Leftrightarrow\delta\mathscr{L}=0 \tag{12}$$

where

$$\mathscr{L}=\tfrac{1}{2}u_{2x}^2-\tfrac{3}{4}auu_x^2-\tfrac{1}{2}v_x^2+\tfrac{1}{24}a^2u^4+\tfrac{1}{4}auv^2+\frac{\alpha}{4v^2}. \tag{13}$$

It is also a canonical Hamiltonian system in Ostrogradsky variables

$$\begin{aligned}
&q_1=u \qquad p_1=-\tfrac{3}{2}auu_x-u_{3x}\\
&q_2=u_x \qquad p_2=u_{2x}\\
&q_3=v \qquad p_3=-v_x
\end{aligned} \tag{14}$$

with the Hamiltonian function

$$h(q,p)=\tfrac{1}{2}p_2^2+\tfrac{1}{2}p_3^2+p_1q_2-p_3q_3+\tfrac{3}{4}aq_1q_2^2-\tfrac{1}{24}a^2q_1^4-\tfrac{1}{4}aq_1q_3^2-\frac{\alpha}{4q_3^2}. \tag{15}$$

In the special case of the fifth-order Sawada–Kotera stationary flow, Ostrogradsky representation (10) turns out to be a generalized Henon–Heiles representation [3].

This work was partially supported by a stypendy AdT/MLH ref 304/31 from Swedish Institute.

References

[1] Ostrogradsky M 1850 *Mem. Acad. St Petersburg* **6** 385
[2] Whittaker E T 1937 *A Treatise on the Analytic Dynamics of Particles and Rigid Bodies* (Cambridge: Cambridge University Press)
[3] Antonowicz M and Rauch-Wojciechowski S 1992 *Phys. Lett.* **163A** 167